1 - 9 Further ODEs reducible to Bessel's ODE

Find a general solution in terms of J_ν and Y_ν . Indicate whether you could also use $J_{-\nu}$ instead of Y_ν. Use the indicated substitution.

1. $x^2 y'' + x y' + (x^2 - 16) y = 0$

```
ClearAll["Global`*"]
```

```
e1 = {x^2 y''[x] + x y'[x] + (x^2 - 16) y[x] == 0}
e2 = DSolve[e1, y, x]
```

```
{(-16 + x^2) y[x] + x y′[x] + x^2 y′′[x] == 0}
```

```
{{y → Function[{x}, BesselJ[4, x] C[1] + BesselY[4, x] C[2]]}}
```

```
e1 /. e2 // FullSimplify
```

```
{{True}}
```

The above answer matches the text's. I believe that **FullSimplify** is needed to check **DSolve** in this case because Bessels are special functions.

3. $9 x^2 y'' + 9 x y' + (36 x^4 - 16) y = 0 \quad (x^2 = z)$

```
ClearAll["Global`*"]
```

```
e1 = {9 x^2 y''[x] + 9 x y'[x] + (36 x^4 - 16) y[x] == 0}
```

```
{(-16 + 36 x^4) y[x] + 9 x y′[x] + 9 x^2 y′′[x] == 0}
```

```
e2 = DSolve[e1, y, x, Assumptions → x^2 → z]
```

```
{{y → Function[{x},
    BesselJ[-2/3, x^2] C[1] Gamma[1/3] + BesselJ[2/3, x^2] C[2] Gamma[5/3]]}}
```

Mathematica demonstrates that its anwer is good.

```
e1 /. e2 // FullSimplify
```

```
{{True}}
```

The PZQ says that the Mathematica answer does not match that of the text.

```
PossibleZeroQ[BesselJ[-2/3, x^2] Gamma[1/3] +
  BesselJ[2/3, x^2] Gamma[5/3] - (BesselJ[2/3, x^2] + BesselY[2/3, x^2])]
False
```

5. $4 x y'' + 4 y' + y = 0 \quad \left(\sqrt{x} = z\right)$

```
ClearAll["Global`*"]
e1 = {4 x y''[x] + 4 y'[x] + y[x] == 0}
e2 = DSolve[e1, y[x], x, Assumptions → √x → z]
{y[x] + 4 y'[x] + 4 x y''[x] == 0}
```

```
{{y[x] → BesselJ[0, √x] C[1] + 2 BesselY[0, √x] C[2]}}
```

The answer above agrees with the text answer, as I interpret it. It appears that C[2] and the 2 factor in the second term need to be combined.

7. $y'' + k^2 x^2 y = 0 \quad \left(y = u \sqrt{x},\ \frac{1}{2} k x^2 = z\right)$

```
ClearAll["Global`*"]
e1 = {y''[x] + k^2 x^2 y[x] == 0}
{k^2 x^2 y[x] + y''[x] == 0}
```

```
e2 = DSolve[e1, y, x, Assumptions → {y[x] → u √x, k x^2 → z}]
```

```
{{y → Function[{x}, C[1] ParabolicCylinderD[-1/2, (-1)^(1/4) √2 √k x] +
     C[2] ParabolicCylinderD[-1/2, (-1)^(3/4) √2 √k x]]}}
```

Mathematica demonstrates that its answer is correct.

```
e1 /. e2 // FullSimplify
{{True}}
```

$$\text{PossibleZeroQ}\left[\text{ParabolicCylinderD}\left[-\frac{1}{2}, (-1)^{1/4}\sqrt{2}\sqrt{k}\,x\right] + \text{ParabolicCylinderD}\left[-\frac{1}{2}, (-1)^{3/4}\sqrt{2}\sqrt{k}\,x\right] - \sqrt{x}\,\text{BesselJ}\left[\frac{1}{4}, \frac{1}{2}k\,x^2\right] - \text{BesselY}\left[\frac{1}{4}, \frac{1}{2}k\,x^2\right]\right]$$

False

It appears that Mathematica's answer does not equal that of the text.

9. $x\,y'' - 5\,y' + x\,y = 0 \quad (y = x^3\,u)$

```
ClearAll["Global`*"]
```

```
e1 = {x y''[x] - 5 y'[x] + x y[x] == 0}
e2 = DSolve[e1, y, x, Assumptions → y[x] → x^3 u]
```

$\{x\,y[x] - 5\,y'[x] + x\,y''[x] == 0\}$

$\{\{y \to \text{Function}[\{x\}, x^3\,\text{BesselJ}[3, x]\,C[1] + x^3\,\text{BesselY}[3, x]\,C[2]]\}\}$

```
e1 /. e2 // FullSimplify
```

{{True}}

The above answer agrees with the text's.

11 - 15 Hankel and modified Bessel functions

11. Hankel functions. Show that the Hankel functions (10) form a basis of solutions of Bessel's equation for any ν.

```
ClearAll["Global`*"]
```

$H_\nu^{(1)}(x) = J_\nu(x) + i\,Y_\nu(x)$
$H_\nu^{(2)}(x) = J_\nu(x) - i\,Y_\nu(x)$

```
e1 = c1 (jν + i yν) + c2 (jν - i yν) == 0
```

$c2\,(j\nu - i\,y\nu) + c1\,(j\nu + i\,y\nu) == 0$

Above: inserted definitions. It is necessary to change the symbols, I suppose Mathematica recognzied the traditional forms of the Bessels.

```
e2 = Expand[e1]
```

$c1\,j\nu + c2\,j\nu + i\,c1\,y\nu - i\,c2\,y\nu == 0$

```
e3 = Collect[e2, {jν, yν}]
```

$(c1 + c2)\,j\nu + (i\,c1 - i\,c2)\,y\nu == 0$

```
e4 = e3 /. (ⅈ c1 - ⅈ c2) → ⅈ (c1 - c2)
(c1 + c2) jν + ⅈ (c1 - c2) yν == 0
```

jν and **yν** are known to be linearly independent. (Multiplying one of them by i will not change their linear independence.) That means that the above equation can only be true if $(c1 + c2)$ and $(c1 - c2)$ are both zero.

```
Solve[(c1 + c2) == 0 && (c1 - c2) == 0, {c1, c2}]
{{c1 → 0, c2 → 0}}
```

The above tells me that the two expressions, which were definitions of the Hankel functions, are linearly independent.